HOMOLOGY GROUPS OF PIPELINE PETRI NETS [1]

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Abstract

We study homology groups of elementary Petri nets for the pipeline systems. We show that the integral homology groups of these nets equal the group of integers in dimensions 0 and 1, and they are zero in other dimensions. We prove that directed homology groups of elementary Petri nets are zero in all dimensions.

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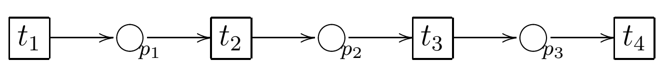
Key words: homology of small categories, trace monoid, asynchronous transition system, elementary Petri net, pipeline, semicubical set.

Introduction

The homology groups of elementary Petri nets were introduced in [1]. In [1], it was built also an algorithm to compute the first homology group of asynchronous systems, based on which we calculated the homology of Petri net of pipeline $\mathfrak{P}_3$, consisting of three transitions. In [2], an algorithm for computing the homology groups of elementary Petri net was developed. An example of calculating the homology groups of elementary Petri net for pipeline $\mathfrak{P}_4$



was considered.

We calculated the homology group for $\mathfrak{P}_n$, with $n = 2, 3, 4, 5$ by the software described in [3]. In these cases, the homology group for the pipelines are $H_0(\mathfrak{P}_n) = H_1(\mathfrak{P}_n) = \mathbb{Z}$, and $H_k(\mathfrak{P}_n) = 0$ for $k \geqslant 2$. There was a conjecture that this is true for all $n \geqslant 2$. In this paper, we prove this conjecture. In addition, the second co-author of this paper has developed the software designed to calculate the directed homology groups of the state spaces. Calculations showed that the directed homology groups of elementary Petri nets $\mathfrak{P}_n$ for $n = 2, 3, 4, 5$ are equal to zero in all dimensions. In this paper, we show that this is true for all $n \geqslant 2$.

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1 Preliminaries

For any category $\mathscr{C}$ denote by $\mathrm{Ob}\,\mathscr{C}$ the class of its objects, $\mathrm{Mor}\,\mathscr{C}$ the class of all morphisms, and $\mathscr{C}^{op}$ the opposite category. For any $A, B \in \mathrm{Ob}\,\mathscr{C}$ the set of all morphisms $A \to B$ in a category $\mathscr{C}$ is denoted by $\mathscr{C}(A, B)$.

1.1 Partial maps

For any sets S and T *a partial map* $f : S \rightharpoonup T$ is a relation $f \subseteq S \times T$ with the following property:

$$(\forall s \in S)(\forall t_1, t_2 \in T)\ (s, t_1) \in f \ \&\ (s, t_2) \in f \ \Rightarrow\ t_1 = t_2 \,.$$

We denote $PSet$ category of sets and partial maps.

For any set S denote $S_* = S \sqcup \{*\}$. Let Set_* be a category of sets of the form S_* where S are arbitrary sets. For any of its objects S_* and T_*, let the set of morphisms $\mathrm{Set}_*(S_*, T_*)$ consists of maps $f : S_* \to T_*$ satisfying the equality $f(*) = *$. It is easy to see that functor $\Phi : PSet \to \mathrm{Set}_*$, defined on objects as $\Phi(S) = S_*$, and on morphisms

$$\Phi(f)(x) = \begin{cases} f(x), & \text{if } f(x) \text{ defined,} \\ *, & \text{if } f(x) \text{ is not defined or } x = *, \end{cases}$$

will provide an isomorphism of categories $PSet$ and Set_*. This isomorphism allows us to identify category $PSet$ with Set_* and we can consider a partial map as the (total) function preserving point $*$.

1.2 Partial action of a monoid

Suppose M be a monoid. The operation in M is denoted by $\cdot_M$. Each monoid M will be considered as a small category that has single object pt_M. The set of morphisms of this category is the set M. In particular, the opposite monoid M^{op} is defined. Product of the elements $\mu_1, \mu_2 \in M$ in this monoid is given by $\mu_1 \cdot_{M^{op}} \mu_2 = \mu_2 \cdot_M \mu_1$. *Partial right action of a monoid M on the set S* is an arbitrary homomorphism $M^{op} \to PSet(S, S)$. In accordance with the above agreement, a partial action will be viewed as a homomorphism $M^{op} \to \mathrm{Set}_*(S_*, S_*)$. It follows that it can be regarded as a functor $M^{op} \to \mathrm{Set}_*$.

1.3 Trace monoids and state space

Let E be a set, and let $I \subseteq E \times E$ be a irreflective symmetric relation on E. Let $M(E, I)$ be a monoid generated by the set E and given by the equalities $ab = ba$ for all $(a, b) \in I$. The monoid $M(E, I)$ is called to be a *free partially commutative monoid* or *trace monoid.* $M(E, I)$ is equal to the quotient monoid $E^*/\equiv$ where E^* is the monoid of words and $\equiv$ is the smallest congruence relation containing a pair of (ab, ba) for all pairs $(a, b) \in I$. *The space of states* is a pair $(S, M(E, I))$, where $M(E, I)$ is a trace monoid with action on a set S.

1.4 The state space of elementary Petri nets

For any elementary Petri net $\mathcal{N} = (P, E, pre, post, s_0)$, we consider a trace monoid $M(E, I)$, generated by the set E with the independence relation

$$(a, b) \in I \ \Leftrightarrow \ (pre(a) \cup post(a)) \cap (pre(b) \cup post(b)) = \emptyset.$$

Suppose $a \in E$. For any subset $s \subseteq P$, with properties

- $pre(a) \subseteq s$,
- $(s \setminus pre(a)) \cap post(a) = \emptyset$,

we denote $s \cdot a = (s \setminus pre(a)) \cup post(a)$. This defines for each $a \in E$ a partial map $s \mapsto s \cdot a$, of the set $\{0, 1\}^P$ in $\{0, 1\}^P$. Here the subsets $s \subseteq P$ are considered as their characteristic functions.

The obtained partial map (corresponding to the relation of the transition from [4]), can be defined as the set of triples (s, a, s') with the property

$$pre(a) \subseteq s \ \& \ post(a) \subseteq s' \ \& \ s \setminus pre(a) = s' \setminus post(a).$$

Using the induction on the length of the trace $\mu = a_1 a_2 \cdots a_n$ by the formula $s \cdot a_1 a_2 \cdots a_n = (s \cdot a_1 a_2 \cdots a_{n-1}) \cdot a_n$, we can define a partial right action of a monoid $M(E, I)$ on $\{0, 1\}^P$. The resulting state space $(\{0, 1\}^P, M(E, I))$ will be called a *state space of elementary Petri nets.*

If we consider only the set of reachable states instead of $\{0, 1\}^P$, then this partial action will define the *space of reachable states of the elementary Petri net.*

2 Homology of semicubical sets and state spaces

Recall the definition of semicubical set and its homology groups [5]. We describe semicubical set corresponding to a state category with the same homology groups.

2.1 Semicubical sets

A *semicubical set* $X = (X_n, \partial_i^{n,\varepsilon})$ is given as a sequence sets X_n, $n \geqslant 0$ with a family of maps $\partial_i^{n,\varepsilon} : X_n \to X_{n-1}$, defined by $1 \leqslant i \leqslant n$, $\varepsilon \in \{0,1\}$, for which the following diagrams

$$\begin{tikzcd} X_n \arrow[r, "\partial_i^{n,\alpha}"] \arrow[d, "\partial_j^{n,\beta}"'] & X_{n-1} \arrow[d, "\partial_{j-1}^{n-1,\beta}"] \\ X_{n-1} \arrow[r, "\partial_i^{n-1,\alpha}"'] & X_{n-2} \end{tikzcd}$$

are commutative for all $n \geqslant 2,\ 1 \leqslant i < j \leqslant n$.

Morphism of semicubical sets $f : X \to \tilde{X}$ is a sequence of maps $f_n : X_n \to \tilde{X}_n$, $n \geqslant 0$, for which the diagrams

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "\partial_i^{n,\varepsilon}"'] & \tilde{X}_n \arrow[d, "\tilde{\partial}_i^{n,\varepsilon}"] \\ X_{n-1} \arrow[r, "f_{n-1}"'] & \tilde{X}_{n-1} \end{tikzcd}$$

are commutative for all $1 \leqslant i \leqslant n$ and $\varepsilon \in \{0,1\}$. If the morphism semicubical sets consist of inclusions $X_n \subseteq \tilde{X}_n$ for all $n \geqslant 0$, then X is called to be a *semicubical subset* of $\tilde{X}$.

2.2 Homology of semicubical sets

Let $X = (X_n, \partial_i^{n,\varepsilon})$ be a semicubical set. Consider the complex

$$0 \leftarrow LX_0 \stackrel{d_1}{\leftarrow} LX_1 \stackrel{d_2}{\leftarrow} LX_2 \stackrel{d_1}{\leftarrow} \cdots ,$$

consisting of free Abelian groups LX_n, $n \geqslant 0$, generated by the sets X_n, and differentials, acting on the basis elements by the formula

$$d_n(\sigma) = \sum_{i=1}^{n} (-1)^i (\partial_i^{n,1}(\sigma) - \partial_i^{n,0}(\sigma)).$$

The homology groups of this complex are called to be *homology groups* $H_n(X)$ *of semicubical set* X, $n \geqslant 0$.

Proposition 2.1 *Suppose that $X = X_1 \cup X_2$ is a union of semicubical subsets. Then there is a long exact sequence of groups*

$$\begin{aligned} 0 \leftarrow H_0(X) \leftarrow H_0(X_1) \oplus H_0(X_2) \leftarrow H_0(X_1 \cap X_2) \leftarrow \cdots \\ \cdots \leftarrow H_n(X) \leftarrow H_n(X_1) \oplus H_n(X_2) \leftarrow H_n(X_1 \cap X_2) \leftarrow \cdots \end{aligned}$$

Proof. Consider the complex $C_n(X) = LX_n$ with differentials $d_n(\sigma) = \sum\limits_{i=1}^{n} (-1)^i (\partial_i^{n,1}(\sigma) - \partial_i^{n,0}(\sigma))$. Consider the exact sequence of complexes associated with the homomorphism $\sigma_1 \oplus \sigma_2 \mapsto \sigma_1 - \sigma_2$,

$$0 \to C_n(X_1 \cap X_2) \xrightarrow{\theta_n} C_n(X_1) \oplus C_n(X_2) \xrightarrow{-} C_n(X_1 \cup X_2) \to 0,$$

where $\theta_n(\sigma) = \sigma \oplus \sigma$ for each $\sigma \in (X_1 \cap X_2)_n$. A long exact sequence corresponding to this short exact sequence, will be required. $\square$

2.3 Homology of state category and semicubical sets

Let $(S, M(E, I))$ be a state space. Consider an arbitrary linear order relation on E. It defines a semicubical set

$$\begin{aligned} Q_n(S, E, I) = \{(s, a_1, \ldots, a_n) \in S \times E^n \mid a_1 < \cdots < a_n \ \&\ s \cdot a_1 \cdots a_n \in S \\ \&\ (a_i, a_j) \in I \text{ for all } 1 \leqslant i < j \leqslant n\}, \quad n \geqslant 0, \end{aligned}$$

with boundary operators

$$\partial_i^{n,\varepsilon}(s, a_1, \ldots, a_n) = (s \cdot a_i^{\varepsilon}, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n),$$

for $1 \leqslant i \leqslant n$, $\varepsilon \in \{0, 1\}$. There $a_i^0 = 1$ and $a_i^1 = a_i$.

Let $(S, M(E,I))$ be a state space. Its *state category* $K(S)$ is defined as a small category with objects $s \in S$. Morphisms $s \xrightarrow{\mu} t$ are defined as triples of elements $s, t \in S$ and $\mu \in M(E,I)$ satisfying $s \cdot \mu = t$. Composition is given by the formula $(t \xrightarrow{\nu} u) \circ (s \xrightarrow{\mu} t) = (s \xrightarrow{\mu\nu} u)$.

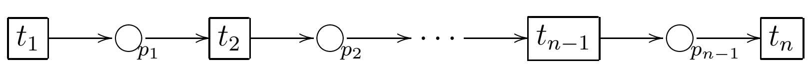
For any elementary Petri net $\mathcal{N}$, we consider the state space and its state category. Let $K(S)$ be a full subcategory of the state category consisting of reachable states. *Homology groups* $H_n(\mathcal{N})$ is defined as homology groups $H_n(K(S))$ of the nerve of the category $K(S)$. According to [2, Corollary 4], $H_n(K(S)) \cong H_n(Q(S,E,I))$, for all $n \geqslant 0$. Therefore, the homology groups of elementary Petri nets $H_n(\mathcal{N})$ are isomorphic to the homology of semicubical sets corresponding to its space of reachable states.

3 State category of the pipeline Petri net and its homology groups

In this section, we will explore the state category of pipeline elementary Petri net and calculate the integral homology groups of this category.

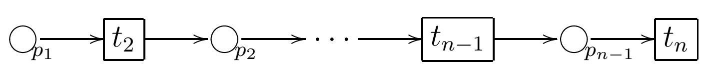
3.1 State category of pipeline elementary Petri net

We consider pipeline Petri net $\mathfrak{P}_n$:



Represent the state category of this net as union of two partially ordered sets, each of which has the least and the greatest element.

Let $\mathcal{N}_n$ be the following Petri net



Denote by $\mathscr{C}_n$ its the state category. Every partially ordered set can be considered as a small category $\mathscr{C}$ such that for any objects $x, y \in \operatorname{Ob}\mathscr{C}$, the set $\mathscr{C}(x,y)$ contains no more than one element, and if the conjunction of $\mathscr{C}(x,y) \neq \emptyset$ and $\mathscr{C}(y,x) \neq \emptyset$ implies $x = y$.

Any state of elementary Petri net $\mathcal{N}_n$ is given by an arbitrary function $f : \{p_1, p_2, \cdots, p_{n-1}\} \to \{0, 1\}$. It follows that the state can be given by a sequence of values $\varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}$, where $\varepsilon_i = f(p_i)$ of f for $1 \leqslant i \leqslant n-1$. Let

S_1 the set of states beginning with $\varepsilon_1 = 1$, and S_0 the set of states, beginning with $\varepsilon_1 = 0$. The set of all states will be equal to $S = S_1 \sqcup S_0$. In general, the elements of S_1 and S_0 are connected by transitions $10\varepsilon_3 \cdots \varepsilon_{n-1} \xrightarrow{t_2} 01\varepsilon_3 \cdots \varepsilon_{n-1}$.

For example, if $n = 4$, then S_1 and S_0 are linearly ordered sets corresponding columns of the diagram:

$$\begin{tikzcd}
111 \arrow[d, "t_4"'] & & 011 \arrow[d, "t_4"] \\
110 \arrow[d, "t_3"'] & & 010 \arrow[d, "t_3"] \\
101 \arrow[d, "t_4"'] \arrow[uurr, dashed, "t_2"'] & & 001 \arrow[d, "t_4"] \\
100 \arrow[uurr, dashed, "t_2"'] & & 000
\end{tikzcd}$$

Transitions between S_1 and S_0 is shown by the dashed arrows.

For any state $x = \varepsilon_1 \cdots \varepsilon_{n-1}$, we introduce a number $|x| = \varepsilon_1 \cdot 2^{n-2} + \varepsilon_2 \cdot 2^{n-3} + \cdots + \varepsilon_{n-1} \cdot 2^0$. For example, $|11\cdots 1| = 2^{n-1} - 1$ and $|00\cdots 0| = 0$. If there is a transition $x \xrightarrow{t_k} y$, then $|x| < |y|$

Lemma 3.1 *Category $\mathscr{C}_n$ is a partially ordered set having the greatest and least elements.*

Proof. By induction, we prove that $\mathscr{C}_n$ is a partially ordered set with the least element of $11\cdots 1$ and the greatest element $00\cdots 0$. Let it proved for $n-1$. Then $\mathscr{C}_{n-1}$ is the partially ordered set with the least and greatest elements. It is easy to see that the full subcategory of $\mathscr{C}_n$ with set of objects S_1 and S_0 are isomorphic to the category $\mathscr{C}_{n-1}$. So, they will be partially ordered sets. We denote these posets by S_1 and S_0. In S_1, a least element of the induction assumption is $11\cdots 1$, and the greatest element equals $10\cdots 0$. In S_0, a least element is $01\cdots 1$, and the greatest equals $00\cdots 0$.

Let $x\mu = y$, $x \in S_1$, $y \in S_0$. Then μ contains the letter t_2. Therefore, there exist μ_1 and μ_2, such that $\mu = \mu_1 t_2 \mu_2$, and μ_1 does not contain t_2. Now, we will rearrange the letters contained in mu_1 with the letter t_2, until we do not meet t_3. For some μ_1 and μ_2, we obtain the decomposition $\mu = \mu_1 t_3 t_2 \mu_2$.

The following step we will perform until we get the decomposition of the form $\mu = t_k \cdots t_2 \mu'$.

Suppose that $\mu = \mu_1 t t_{k-1} \cdots t_2$, for some $k \geqslant 3$ and $t \in E$. We describe the action taken by us in each of the following possible cases:

(i) $t = t_k \Rightarrow$ increase k by 1,

(ii) $t = t_i$, for some $i > k \Rightarrow$ swap t with t_{k-1}, t_{k-2}, $\cdots$, t_2.

For $i \leqslant k-1$, a decomposition of the form $\mu = \mu_1 t_i t_{k-1} \cdots t_2$ impossible, since an element $x' t_{k-1} \cdots t_2$ is defined if and only if $x' = 11 \cdots 10 \varepsilon_k \cdots \varepsilon_{n-1}$. Such element x' can not be obtained after the action of t_i for $2 \leqslant i \leqslant k-1$. It follows that the iteration of the described action will lead to the decomposition $\mu = t_k \cdots t_2 \mu'$.

Now we will prove that $x\mu = x\nu = y \in S$ implies $\mu = \nu$, which implies that $\mathscr{C}_n$ is a preordered set.

If $x\mu = x\nu = y \in S_0$ and $y \in S_0$, then $\mu = \nu$ because of S_0 is a partially ordered set. The same is true if $x\mu = x\nu = y \in S_1$ and $y \in S_1$. There are no $\mu \in M(E, I)$ and $x \in S_0$, for which $x\mu \in S_1$.

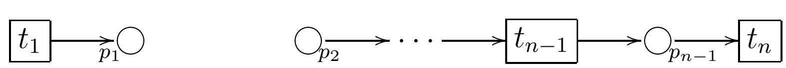
Now consider the case $x\mu = x\nu = y \in S_0$, $x \in S_1$. In this case, there exist decompositions $\mu = t_k \cdots t_2 \mu'$ and $\nu = t_m \cdots t_2 \nu'$. Since $x t_k t_{k-1} \cdots t_2 \mu' \in S_0$, then $x = 11 \cdots 10 \varepsilon_{k+1} \cdots \varepsilon_{n-1}$, for some $\varepsilon_{k+1}, \cdots, \varepsilon_{n-1} \in \{0, 1\}$. Hence $m = k$. We get the equality

$$x t_k \cdots t_2 \mu' = x t_k \cdots t_2 \nu'$$

Since $x t_k \cdots t_2 \in S_0$ and S_0 – partially ordered set it follows from this equation $\mu' = \nu'$. Therefore, $\mu = \nu$ and $\mathscr{C}_n$ – preordered set. If for some $x = \varepsilon_1 \cdots \varepsilon_{n-1}$, $y = \delta_1 \cdots \delta_{n-1}$, there exists $\mu \neq 1$, which $x\mu = y$, then $|x| > |y|$. Therefore, from $x\mu = y$ and $y\nu = x$ will follow the $\mu = \nu = 1$. Hence otnshenie preorder will antisymmetric and $\mathscr{C}_n$ is partially ordered set. $\square$

3.2 The homology groups of pipeline

Let $\mathfrak{P}_n$ – elementary net pipeline. Delete event t_1, we have a network $\mathcal{N}_n$. Remove from $\mathfrak{P}_n$ event t_2, we get the following elementary network:



We denote it by $\mathcal{N}'_n$.

Proposition 3.2 *Semicubical set $Q(\mathfrak{P}_n)$ is the union of subsets semicubical $Q(\mathcal{N}_n) \cup Q(\mathcal{N}'_n)$. Intersection of $Q(\mathcal{N}_n) \cap Q(\mathcal{N}'_n)$ semicubical be set with a two components, each of which is isomorphic $Q(\mathcal{N}_{n-1})$.*

Proof. There are $Q_0(\mathfrak{P}_n) = Q_0(\mathcal{N}_n) \cup Q_0(\mathcal{N}'_n)$, because all of these sets are $S = \{0,1\}^{n-1}$. If $k \geqslant 1$, then since t_1 and t_2 dependent, they can not belong a set of mutually independent events $(e_1, \cdots, e_m)$. This means that for every $(s, e_1, \cdots, e_m) \in Q_m(\mathfrak{P}_n)$ will be a $t_1 \notin \{e_1, \cdots, e_m\}$ or $t_2 \notin \{e_1, \cdots, e_m\}$, where $(s, e_1, \cdots, e_m) \in Q_m(\mathcal{N}_n) \cup Q_m(\mathcal{N}'_n)$. Because of $(s, e_1, \cdots, e_m) \in Q_m(\mathcal{N}_n)$ follows $(s \cdot e_i^{\varepsilon}, e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_m) \in Q_{m-1}(\mathcal{N}_n)$ and from $(s, e_1, \cdots, e_m) \in Q_m(\mathcal{N}'_n)$ follows $(s \cdot e_i^{\varepsilon}, e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_m) \in Q_{m-1}(\mathcal{N}'_n)$, attachments commute with the boundary operators. So, $Q(\mathfrak{P}_n)$ equals the union its cubic subsets of $Q(\mathcal{N}_n)$ and $Q(\mathcal{N}'_n)$. □

Theorem 3.3 *$H_0(\mathfrak{P}_n) = H_1(\mathfrak{P}_n) = \mathbb{Z}$, and $H_k(\mathfrak{P}_n) = 0$ at $k > 1$.*

Proof. By Proposition 3.2, $Q(\mathfrak{P}_n) = Q(\mathcal{N}_n) \cup Q(\mathcal{N}'_n)$. Apply Proposition 2.1. By Lemma 3.1 category network states $\mathcal{N}_n$ has initial and terminal object. Let $\mathbb{I} = \{0,1\}$ – partially ordered set consisting of the elements of 0 and 1, with the usual order relation $0 < 1$. It is easy to see that the category of network conditions $\mathcal{N}'_n$ is isomorphic to the product of $\mathbb{I} \times \mathscr{C}_{n-2}$. Therefore it has initial and terminal facilities too. From this follow isomorphisms $H_k(Q(\mathcal{N}_n)) = H_k(Q(\mathcal{N}'_n)) = 0$, for $k > 0$. The exact sequence 2.1 leads to the isomorphisms $H_k(Q(\mathfrak{P}_n)) \cong H_{k-1}(Q(\mathcal{N}_n) \cap Q(\mathcal{N}'_n))$, for $k \geqslant 2$. Of Proposition 3.2 follows $Q(\mathcal{N}_n) \cap Q(\mathcal{N}'_n) \cong Q(\mathcal{N}_{n-1}) \sqcup Q(\mathcal{N}_{n-1})$. We get $H_k(Q(\mathfrak{P}_n)) \cong H_{k-1}(Q(\mathcal{N}_{n-1})) \oplus H_{k-1}(Q(\mathcal{N}_{n-1})) = 0$, where $k \geqslant 2$. Consequently, $H_k(\mathfrak{P}_n) = 0$ for $k \geqslant 2$. Of Proposition 2.1 also obtain accurate sequence

$$
\begin{aligned}
0 \leftarrow H_0(Q(\mathfrak{P}_n)) \leftarrow H_0(Q(\mathcal{N}_n)) \oplus H_0(Q(\mathcal{N}'_n)) \\
\leftarrow H_0(Q(\mathcal{N}_n) \cap Q(\mathcal{N}'_n)) \leftarrow H_1(Q(\mathfrak{P}_n)) \leftarrow 0.
\end{aligned}
$$

Group H_0 freely generated connected components semicubical sets. Homomorphism

$$H_0(\theta) : H_0(Q(\mathcal{N}_n) \cap Q(\mathcal{N}'_n)) \to H_0(Q(\mathcal{N}_n)) \oplus H_0(Q(\mathcal{N}'_n))$$

induced chain homomorphism

$$\theta_k : C_k(Q(\mathcal{N}_n) \cap Q(\mathcal{N}'_n)) \to C_k(Q(\mathcal{N}_n)) \oplus C_k(Q(\mathcal{N}'_n)),$$

defined by $\sigma \in (Q(\mathcal{N}_n) \cap Q(\mathcal{N}'_n))_k$ by the formula $\theta_k(\sigma) = \sigma \oplus \sigma$. This implies that $H_0(\theta)$ acts on the homology classes of the formula $H_0(\theta)(cls(\sigma)) = cls(\sigma) \oplus cls(\sigma)$. Because these classes homology are connected components semicubical sets, then $H_0(\theta)$ is a homomorphism $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, given by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. But $H_1(\mathfrak{P}_n)$ is isomorphic to kernel of the homomorphism $H_0(\theta)$, and $H_0(\mathfrak{P}_n)$ – cokernel. Hence, for example, with a cast of this matrix to Smith normal form, we obtain $H_0(\mathfrak{P}_n) = H_1(\mathfrak{P}_n) = \mathbb{Z}$. □

4 Directed homology

We present supporting information on the homology groups Goubault semicubical sets. We study the properties of directed homology category states and their interpretation in dimension 0. Calculate the direction of homology Petri net pipeline.

4.1 Homology groups of Goubault

Let $X = (X_n, \partial_i^{n,\varepsilon})$ semicubical set. For all $\varepsilon \in \{0, 1\}$, consider the chain complex of abelian groups $C_n(X) = L(X_n)$ with the differentials

$$0 \leftarrow C_0(X) \xleftarrow{d_1^\varepsilon} C_1(X) \leftarrow \cdots C_{n-1}(X) \xleftarrow{d_n^\varepsilon} C_n(X) \leftarrow \cdots,$$

where $d_n^\varepsilon(\sigma) = \sum\limits_{i=1}^{n} (-1)^i \partial_i^{n,\varepsilon}(\sigma)$.

Homology groups $H_n^\varepsilon(X)$ of the complex called *homology groups Goubault* semicubical set X. They were studied in [6].

Groups $H_n^0(X)$ are called *initial*, and $H_n^1(X)$ – *final* homology groups Goubault.

Proposition 4.1 *Let $X = X_1 \cup X_2$ – union semicubical subsets $X_1 \subseteq X$ and $X_2 \subseteq X$. Then for every $\varepsilon \in \{0, 1\}$ there is a long exact sequence*

$$\begin{aligned} 0 \leftarrow H_0^\varepsilon(X) \leftarrow H_0^\varepsilon(X_1) \oplus H_0^\varepsilon(X_2) \leftarrow H_0^\varepsilon(X_1 \cap X_2) \leftarrow \cdots \\ \cdots \leftarrow H_n^\varepsilon(X) \leftarrow H_n^\varepsilon(X_1) \oplus H_n^\varepsilon(X_2) \leftarrow H_n^\varepsilon(X_1 \cap X_2) \leftarrow \cdots \end{aligned}$$

Proof repeats almost verbatim the proof of Proposition 2.1.

4.2 Directed homology groups of the state category

Homology groups of a small category $\mathscr{C}$ with coefficients in the functor $F : \mathscr{C} \to \mathrm{Ab}$ is defined as values $\varinjlim_n^{\mathscr{C}} F$ of left derived functors of $\varinjlim^{\mathscr{C}} : \mathrm{Ab}^{\mathscr{C}} \to \mathrm{Ab}$ on F.

Let $(S, M(E,I))$ be a state space, and $K(S)$ be its the state category. Consider the functors $\Delta^0 \mathbb{Z} : K(S) \to \mathrm{Ab}$ and $\Delta^1 \mathbb{Z} : K(S)^{op} \to \mathrm{Ab}$, taking on the objects the constant values $\Delta^0 \mathbb{Z}(s) = \Delta^1 \mathbb{Z}(s) = \mathbb{Z}$. And on the morphisms defined by the formulas

$$\Delta^\varepsilon \mathbb{Z}(s \xrightarrow{\mu} s') = \begin{cases} 1_s, & \mu = 1, \\ 0, & \mu \neq 1, \end{cases}$$

Definition 4.1 *Directed homology groups of the state space defined by the formulas*

$$H_n^0(S, M(E,I)) =_{def} \varinjlim_n^{K(S)} \Delta^0 \mathbb{Z}, \quad H_n^1(S, M(E,I)) =_{def} \varinjlim_n^{K(S)^{op}} \Delta^1 \mathbb{Z}.$$

For any elementary Petri nets $\mathcal{N}$ its homology groups $H_n^\varepsilon(\mathcal{N})$ defined as the homology groups of its state space.

Proposition 4.2 *For elementary Petri nets $\mathcal{N}_n$ obtained from the pipeline Petri nets $\mathfrak{P}_n$ by deleting the transition t_1, homology groups equal*

$$H_k^\varepsilon(\mathcal{N}_n) = \begin{cases} \mathbb{Z}, & \text{if } k = 0 \\ 0, & \text{if } k > 0. \end{cases}$$

Proof. By Lemma 3.1, the state category $\mathscr{C}_n$ of Petri net $\mathcal{N}_n$ has an initial and terminal objects. If a small category has a terminal object, then the colimit functor on the category equals to the value of this functor on the terminal object. Hence $H_k^\varepsilon(\mathcal{N}_n) = \varinjlim_k \Delta^\varepsilon \mathbb{Z} = 0$ for $k > 0$ and $H_0^\varepsilon(\mathcal{N}_n) = \varinjlim \Delta^\varepsilon \mathbb{Z} = \mathbb{Z}$. $\square$

According to [2, Corollary 5] for any linear order on E, there are isomorphisms

$$\varinjlim_n^{K(S)} \Delta^0 \mathbb{Z} \cong H_n^0(Q(S,E,I)), \quad \varinjlim_n^{K(S)^{op}} \Delta^1 \mathbb{Z} \cong H_n^1(Q(S,E,I)). \tag{1}$$

Hence we get the following interpretation of directed homology for a state space in dimension 0. State s is called to be a *deadlock*, if there is no $a \in E$

satisfying $s \cdot a \in S$. It is called a *sender*, if there is no such a pair $s' \in S$ and $a \in E$ for which $s' \cdot a = s$. In the state category, deadlock is an object s which has not a morphisms $\alpha \neq 1_s$ with $dom\alpha = s$. A sender s has not $\alpha \neq 1_s$ such that $cod\alpha = s$.

Proposition 4.3 *The group $H_0^0(S, M(E,I))$ is isomorphic to the free abelian group generated by deadlocks, and $H_0^1(S, M(E,I))$ generated by senders.*

PROOF. It follows from [2, Corollary 5] that $H_0^0(S, M(E,I))$ is isomorphic to the cokernel of $d_1^0 : LQ_1(S,E,I) \to LQ_0(S,E,I)$ defined as $d_1^0(s,a) = s$. Cokernel is generated by the set obtained by the identification with 0 of all $s \in S$ for which there are $a \in E$ satisfying $s \cdot a \in S$. It is removed from the set S all objects which are no deadlocks. The remaining set generates the cokernel. It is proved similarly that $H_0^1(S, M(E,I))$ generated by senders.$\Box$

EXAMPLE 4.2 *State category of pipeline elementary Petri nets has no senders or deadlocks, hence $H_0^0(\mathfrak{P}_n) = H_0^1(\mathfrak{P}_n) = 0$.*

4.3 Directed homology groups of pipeline

According to the above example, groups $H_0^0(\mathfrak{P}_n)$ and $H_0^1(\mathfrak{P}_n)$ equal zero. The following statement shows that $H_k^0(\mathfrak{P}_n)$ and $H_k^1(\mathfrak{P}_n)$ equal 0 for all $k \geqslant 0$.

Theorem 4.4 *$H_k^\varepsilon(\mathfrak{P}_n) = 0$, for all $n \geqslant 2$, $k \geqslant 0$ and $\varepsilon \in \{0,1\}$.*

PROOF. Above, we found that the semicubical set $Q(\mathfrak{P}_n)$ is equal to union of semicubical sets $Q(\mathcal{N}_n)$ and $Q(\mathcal{N}'_n)$. The state categories of $\mathcal{N}_n$ and $\mathcal{N}'_n$ have the greatest and the least elements. This means that $H_k^\varepsilon(\mathcal{N}_n) = H_k^\varepsilon(\mathcal{N}'_n) = 0$ for $k > 0$, and $H_0^\varepsilon(\mathcal{N}_n) = H_k^\varepsilon(\mathcal{N}'_n) = \mathbb{Z}$. By the suggestion of 3.2, $Q(\mathcal{N}_n) \cap Q(\mathcal{N}'_n) \cong Q(\mathcal{N}_{n-1}) \sqcup Q(\mathcal{N}_{n-1})$. It follows that the exact sequence of the fragment suggests 4.1

$$\leftarrow H_{k-1}^\varepsilon(Q(\mathcal{N}_n) \cap Q(\mathcal{N}'_n)) \leftarrow H_k^\varepsilon(Q(\mathfrak{P}_n)) \leftarrow H_k^\varepsilon(Q(\mathcal{N}_n)) \oplus H_k^\varepsilon(Q(\mathcal{N}'_n)) \leftarrow$$

leads to the equations $H_k^\varepsilon(Q(\mathfrak{P}_n)) = 0$ for $k \geqslant 2$. In addition, there is an exact sequence

$$0 \leftarrow H_0^\varepsilon(Q(\mathfrak{P}_n)) \leftarrow H_0^\varepsilon(Q(\mathcal{N}_n)) \oplus H_0^\varepsilon(Q(\mathcal{N}'_n)) \xleftarrow{H_0^\varepsilon(\theta)} H_0^\varepsilon(Q(\mathcal{N}_n) \cap Q(\mathcal{N}'_n)) \leftarrow H_1^\varepsilon(Q(\mathfrak{P}_n)) \leftarrow 0$$

By example 4.2 and isomorphisms (1), we have $H_0^{\varepsilon}(Q(\mathfrak{P}_n)) = 0$. We obtain an epimorphism $\mathbb{Z} \oplus \mathbb{Z} \stackrel{H_0^{\varepsilon}(\theta)}{\to} \mathbb{Z} \oplus \mathbb{Z}$. But $\mathbb{Z} \oplus \mathbb{Z}$ is projective object in the category of Abelian groups. It follows that there exists a homomorphism γ such that $H_0^{\varepsilon}(\theta) \circ \gamma = 1_{\mathbb{Z}\oplus\mathbb{Z}}$. This means that the determinant of $H_0^{\varepsilon}(\theta)$ is invertible where the kernel $H_0^{\varepsilon}(\theta)$ equally zero. Consequently, $H_1^{\varepsilon}(Q(\mathfrak{P}_n)) = 0$. We have proved that $H_k^{\varepsilon}(Q(\mathfrak{P}_n)) = 0$ for all $k \geqslant 0$. Isomorphisms (1) give $H_k^{\varepsilon}(\mathfrak{P}_n) \cong H_k^{\varepsilon}(Q(\mathfrak{P}_n))$. □

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